

KORTEWEG - DE VRIES SOLITONS IN RELATIVISTIC HYDRODYNAMICS

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Abstract

In a previous work, assuming that the nucleus can be treated as a perfect fluid, we have studied the propagation of perturbations in the baryon density. For a given equation of state we have derived a Korteweg - de Vries (KdV) equation from Euler and continuity equations in non-relativistic hydrodynamics. Here, using a more general equation of state, we extend our formalism to relativistic hydrodynamics.

I. INTRODUCTION

Long ago [1] it was suggested that Korteweg - de Vries solitons might be formed in the nuclear medium. In a previous work [2] we have updated the early works on the subject introducing a realistic equation of state (EOS) for nuclear matter. We have found that these solitary waves can indeed exist in the nuclear medium, provided that derivative couplings between the nucleon and the vector field are included. These couplings lead to an energy density which depends on the Laplacian of the baryon density. For this class of equations of state, which is quite general (as pointed out in [3, 4]), perturbations on the nuclear density can propagate as a pulse without dissipation.

During the analysis of several realistic nuclear equations of state, we realized that, very often the speed of sound c_s is in the range $0.15 - 0.25$. Compared to the speed of light these values are not large but not very small either. This suggests that, even for slowly moving nuclear matter, relativistic effects might be sizeable. This concern motivates the extension of the formalism presented in [2] to relativistic hydrodynamics.

In the next section we review the most relevant equations writing them in an appropriate form for the subsequent manipulations. In the following section we discuss three models for the equation of state and in the next section we derive the KdV equation for the proposed models and present their solutions. In the final section we present our conclusions.

II. RELATIVISTIC HYDRODYNAMICS

In this section we review the main expressions of relativistic hydrodynamics. In natural units ($c = 1$) the velocity four vector u^ν is defined as:

$$u^\nu = (u^0, \vec{u}) = (\gamma, \gamma \vec{v}) \quad (1)$$

where γ is the Lorentz contraction factor given by:

$$\gamma = (1 - v^2)^{-1/2} \quad (2)$$

The velocity field of matter is $\vec{v} = \vec{v}(t, x, y, z)$ and thus $u^\nu u_\nu = 1$. The energy-momentum tensor is, as usual, given by:

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu - pg_{\mu\nu} \quad (3)$$

where ε and p are the energy density and pressure respectively. Energy-momentum conservation is ensured by:

$$\partial_\nu T_\mu{}^\nu = 0 \quad (4)$$

The projection of (4) onto a direction perpendicular to u^μ gives us the relativistic version of Euler equation [5, 6, 7]:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{(\varepsilon + p)\gamma^2} \left(\vec{\nabla} p + \vec{v} \frac{\partial p}{\partial t} \right) \quad (5)$$

The relativistic version of the continuity equation for the baryon number is [5, 6, 7]:

$$\partial_\nu j_B{}^\nu = 0 \quad (6)$$

Since $j_B{}^\nu = u^\nu \rho_B$ the above equation reads

$$\frac{\partial}{\partial t}(\rho_B \gamma) + \vec{\nabla} \cdot (\rho_B \gamma \vec{v}) = 0 \quad (7)$$

The enthalpy per nucleon is given by [6]:

$$dh = T ds + V dp \quad (8)$$

where $V = 1/\rho_B$ is the specific volume. For a perfect fluid ($ds = 0$) the equation above becomes $dp = \rho_B dh$ and consequently:

$$\vec{\nabla} p = \rho_B \vec{\nabla} h, \quad \frac{\partial p}{\partial t} = \rho_B \frac{\partial h}{\partial t} \quad (9)$$

Inserting (9) in (5) we find:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\rho_B}{(\varepsilon + p)\gamma^2} \left(\vec{\nabla} h + \vec{v} \frac{\partial h}{\partial t} \right) \quad (10)$$

Recalling the Gibbs relation [8]:

$$\varepsilon - Ts + p = \mu_B \rho_B \quad (11)$$

and considering the case where $T = 0$ we obtain:

$$\varepsilon + p = \mu_B \rho_B \quad (12)$$

where ε , p , μ_B and ρ_B are the energy density, pressure, baryochemical potential and baryon density respectively. Inserting (2) and (12) into (10) we obtain:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{(1 - v^2)}{\mu_B} \left(\vec{\nabla} h + \vec{v} \frac{\partial h}{\partial t} \right) \quad (13)$$

We close this section comparing the relativistic and non-relativistic versions of the Euler and continuity equations. The latter were presented in [2]:

$$\frac{\partial \rho_B}{\partial t} + \vec{\nabla} \cdot (\rho_B \vec{v}) = 0 \quad (14)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \left(\frac{1}{M} \right) \vec{\nabla} h \quad (15)$$

and the former are (7) and (13):

$$\frac{\partial}{\partial t}(\rho_B \gamma) + \vec{\nabla} \cdot (\rho_B \gamma \vec{v}) = 0 \quad (16)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \frac{(1 - v^2)}{\mu_B} \left(\vec{\nabla} h + \vec{v} \frac{\partial h}{\partial t} \right) \quad (17)$$

The two pairs are similar. The differences are only in the γ factors and in the last term of (17), where the appearance of time derivative reflects the symmetry between space and time. Since the enthalpy per nucleon may also be written as [2, 9]:

$$h = \frac{\partial \varepsilon}{\partial \rho_B} \quad (18)$$

it becomes clear that the “force” on the right hand side of (15) and (17) will be ultimately determined by the equation of state, i.e., by the function $\varepsilon(\rho_B)$.

III. EQUATION OF STATE

Equations (15) and (17) contain the gradient of the derivative of the energy density. If ε contains a Laplacian of ρ_B , i.e., $\varepsilon \propto \dots + \dots \nabla^2 \rho_B + \dots$, then (15) and (17) will have a cubic derivative with respect to the space coordinate, which will give rise to the Korteweg-de Vries equation for the baryon density. The most popular relativistic mean field models do not have higher derivative terms and, even if they have at the start, these terms are usually neglected during the calculations.

In [2] we have added a new derivative term to the usual non-linear QHD [10], given by

$$\mathcal{L}_{\mathcal{M}} \equiv \frac{g_v}{m_v^2} \bar{\psi} (\partial_\nu \partial^\nu V_\mu) \gamma^\mu \psi \quad (19)$$

where, as usual, the degrees of freedom are the baryon field ψ , the neutral scalar meson field ϕ and the neutral vector meson field V_μ , with the respective couplings and masses. The new term is designed to be small in comparison with the main baryon - vector meson interaction

term $g_v \bar{\psi} \gamma_\mu V^\mu \psi$. Following the standard steps of the mean field formalism we have arrived at the following expression for the energy density [2]:

$$\begin{aligned} \varepsilon = & \frac{g_v^2}{2m_v^2} \rho_B^2 + \frac{m_s^2}{2} \left[\frac{(M^* - M)}{g_s} \right]^2 + \frac{\eta}{(2\pi)^3} \int_0^{k_F} d^3k (\vec{k}^2 + M^{*2})^{1/2} + \frac{b}{3g_s^3} (M^* - M)^3 \\ & + \frac{c}{4g_s^4} (M^* - M)^4 + \frac{g_v^2}{m_v^4} \rho_B \nabla^2 \rho_B \end{aligned} \quad (20)$$

where η is the baryon spin-isospin degeneracy factor, M^* stands for the nucleon effective mass (given by $M^* \equiv M - g_s \phi_0$) and the constants b , c , g_s and g_v were taken from [10]. Although Eq. (20) was obtained with the help of a specific Lagrangian taken from [10] and a prototype Laplacian interaction (19), the above form of the energy density follows quite naturally from an approach based on the density functional theory [4], for a wide variety of underlying Lagrangians.

We now follow the treatment developed in [1, 2, 9] to obtain the Korteweg-de Vries equation in one dimension through the combination of (16) and (17). With the help of (20) we first calculate the energy per nucleon given by $E = \varepsilon / \rho_B$. We next perform a Taylor expansion of E around the equilibrium density ρ_0 up to second order:

$$E(\rho_B) = E(\rho_0) + \frac{1}{2} \left(\frac{\partial^2 E}{\partial \rho_B^2} \right)_{\rho_B = \rho_0} (\rho_B - \rho_0)^2 \quad (21)$$

where the first order term vanishes because of the saturation condition:

$$\frac{\partial}{\partial \rho_B} \left(\frac{\varepsilon}{\rho_B} - M \right)_{\rho_B = \rho_0} = \left(\frac{\partial E}{\partial \rho_B} \right)_{\rho_B = \rho_0} = 0 \quad (22)$$

We arrive at (for more details see [2]):

$$E(\rho_B) = E(\rho_0) + \left(\frac{g_v^2}{m_v^4} \right) (\nabla^2 \rho_B) + \frac{1}{2} \frac{M c_s^2}{\rho_0^2} (\rho_B - \rho_0)^2 \quad (23)$$

The enthalpy per nucleon may also be written as [9]:

$$h = E + \rho_B \frac{\partial E}{\partial \rho_B} \quad (24)$$

Using (23) to evaluate (24) and its derivatives we find:

$$\vec{\nabla} h = \frac{3M c_s^2}{\rho_0^2} \rho_B \vec{\nabla} \rho_B - \frac{2M c_s^2}{\rho_0} \vec{\nabla} \rho_B + \frac{g_v^2}{m_v^4} \vec{\nabla} (\vec{\nabla}^2 \rho_B) \quad (25)$$

and

$$\frac{\partial h}{\partial t} = \frac{3M c_s^2}{\rho_0^2} \rho_B \frac{\partial \rho_B}{\partial t} - \frac{2M c_s^2}{\rho_0} \frac{\partial \rho_B}{\partial t} + \frac{g_v^2}{m_v^4} \frac{\partial}{\partial t} (\vec{\nabla}^2 \rho_B) \quad (26)$$

The expressions (25) and (26), from now on referred to as model I, will be inserted into (17) as it will be seen in the next section.

We shall now consider a more general expression for the energy density

$$\varepsilon = \alpha_1 \rho_B + \alpha_2 \rho_B^2 + \alpha_3 \rho_B^3 + \beta \rho_B \vec{\nabla}^2 \rho_B \quad (27)$$

where α_i and β are constants. This Ansatz is similar to the energy density used in [1, 2, 9] and is consistent with the EOS obtained with the approach based on the density functional theory [3, 4]. Let's assume that (27) is an appropriate model for nuclear matter and that it satisfies the saturation condition (22). This will be our model II. Once again we calculate the energy per nucleon $E = \varepsilon/\rho_B$ then Taylor expand it around the equilibrium density ρ_0 up to second order (21) and find

$$E(\rho_B) = \alpha_1 + \alpha_2 \rho_0 + 2\alpha_3 \rho_0^2 + \alpha_3 \rho_B^2 - 2\alpha_3 \rho_B \rho_0 + \beta \vec{\nabla}^2 \rho_B \quad (28)$$

Now, using (28) to evaluate (24) and its derivatives we find:

$$\vec{\nabla} h = 6\alpha_3 \rho_B \vec{\nabla} \rho_B - 4\alpha_3 \rho_0 \vec{\nabla} \rho_B + \beta \vec{\nabla}(\vec{\nabla}^2 \rho_B) \quad (29)$$

and

$$\frac{\partial h}{\partial t} = 6\alpha_3 \rho_B \frac{\partial \rho_B}{\partial t} - 4\alpha_3 \rho_0 \frac{\partial \rho_B}{\partial t} + \beta \frac{\partial}{\partial t}(\vec{\nabla}^2 \rho_B) \quad (30)$$

In model III we consider hadronic matter at arbitrary constant baryon density, but now no saturation condition is imposed. This last choice is motivated by a future study of dense stars. In this case we calculate the enthalpy directly from (27) and (18) obtaining the following expressions for the derivatives:

$$\vec{\nabla} h = 6\alpha_3 \rho_B \vec{\nabla} \rho_B + 2\alpha_2 \vec{\nabla} \rho_B + \beta \vec{\nabla}(\vec{\nabla}^2 \rho_B) \quad (31)$$

and

$$\frac{\partial h}{\partial t} = 6\alpha_3 \rho_B \frac{\partial \rho_B}{\partial t} + 2\alpha_2 \frac{\partial \rho_B}{\partial t} + \beta \frac{\partial}{\partial t}(\vec{\nabla}^2 \rho_B) \quad (32)$$

IV. THE KDV EQUATION

In this section we repeat the steps developed in [1, 2]. We restrict ourselves to the one dimensional case (x, t) and introduce dimensionless variables for the baryon density and velocity:

$$\hat{\rho} = \frac{\rho_B}{\rho_0} , \quad \hat{v} = \frac{v}{c_s} \quad (33)$$

We next define the “stretched coordinates” ξ and τ as in [1, 9, 11]:

$$\xi = \sigma^{1/2} \frac{(x - c_s t)}{R}, \quad \tau = \sigma^{3/2} \frac{c_s t}{R} \quad (34)$$

where R is a size scale and σ is a small ($0 < \sigma < 1$) expansion parameter chosen to be [11]:

$$\sigma = \frac{|u - c_s|}{c_s} \quad (35)$$

where u is the propagation speed of the perturbation in question. We then expand (33) around the equilibrium values:

$$\hat{\rho} = 1 + \sigma \rho_1 + \sigma^2 \rho_2 + \dots \quad (36)$$

$$\hat{v} = \sigma v_1 + \sigma^2 v_2 + \dots \quad (37)$$

After the expansion above (16) and (17) will contain power series in σ (in practice we go up to σ^2). Since the coefficients in these series are independent of each other we get a set of equations, which, when combined, lead to the KdV equation for ρ_1 :

$$\frac{\partial \rho_1}{\partial \tau} + \left(\frac{3}{2} + \frac{\Phi \rho_0^2}{2\mu_B c_s^2} - c_s^2 \right) \rho_1 \frac{\partial \rho_1}{\partial \xi} + \left(\frac{\omega \rho_0}{2\mu_B c_s^2 R^2} \right) \frac{\partial^3 \rho_1}{\partial \xi^3} = 0 \quad (38)$$

with the condition

$$\frac{(\Phi \rho_0 + \phi) \rho_0}{\mu_B c_s^2} = 1 \quad (39)$$

and where

$$\Phi \equiv \begin{cases} 6\alpha_3 & \text{models II and III} \\ \frac{3Mc_s^2}{\rho_0^2} & \text{model I} \end{cases} \quad (40)$$

$$\phi \equiv \begin{cases} 2\alpha_2 & \text{model III} \\ -4\alpha_3 \rho_0 & \text{model II} \\ \frac{-2Mc_s^2}{\rho_0} & \text{model I} \end{cases} \quad (41)$$

$$\omega \equiv \begin{cases} \beta & \text{models II and III} \\ \frac{g_v^2}{m_v^4} & \text{model I} \end{cases} \quad (42)$$

The equation (38) has a well known soliton solution. We may rewrite the last equation back in the $x - t$ space obtaining a KdV-like equation for $\hat{\rho}_1$ with the following analytical solitonic solution:

$$\hat{\rho}_1(x, t) = \frac{3(u - c_s)}{c_s \left(\frac{3}{2} + \frac{\Phi \rho_0^2}{2\mu_B c_s^2} - c_s^2 \right)} \operatorname{sech}^2 \left[\sqrt{\frac{\mu_B c_s (u - c_s)}{2w \rho_0}} (x - ut) \right] \quad (43)$$

where $\hat{\rho}_1 \equiv \sigma \rho_1$. This solution is a bump which propagates with speed u , without dissipation and preserving its shape. The expressions given by (38), (39) and (43) depend on the choices given by (40), (41) and (42).

In model I (MQHD), the constraint (39) implies that $\mu_B = M$ and the general equation (38) becomes:

$$\frac{\partial \rho_1}{\partial \tau} + (3 - c_s^2) \rho_1 \frac{\partial \rho_1}{\partial \xi} + \left(\frac{g_v^2 \rho_0}{2M c_s^2 m_v^4 R^2} \right) \frac{\partial^3 \rho_1}{\partial \xi^3} = 0 \quad (44)$$

with the solution given by:

$$\hat{\rho}_1(x, t) = \frac{3}{(3 - c_s^2)} \frac{(u - c_s)}{c_s} \operatorname{sech}^2 \left[\frac{m_v^2}{g_v} \sqrt{\frac{(u - c_s) c_s M}{2\rho_0}} (x - ut) \right] \quad (45)$$

As a consistency check we take the non-relativistic limit, which, in this case, means taking a small sound speed $c_s^2 \rightarrow 0$. In this limit $(3 - c_s^2) \cong 3$ and (44) and (45) coincide the results previously obtained in [2]:

$$\frac{\partial \rho_1}{\partial \tau} + 3\rho_1 \frac{\partial \rho_1}{\partial \xi} + \left(\frac{g_v^2 \rho_0}{2M c_s^2 m_v^4 R^2} \right) \frac{\partial^3 \rho_1}{\partial \xi^3} = 0 \quad (46)$$

and

$$\hat{\rho}_1(x, t) = \frac{(u - c_s)}{c_s} \operatorname{sech}^2 \left[\frac{m_v^2}{g_v} \sqrt{\frac{(u - c_s) c_s M}{2\rho_0}} (x - ut) \right] \quad (47)$$

In the limit where c_s is large the factor $3/(3 - c_s^2)$ will enhance the soliton amplitude with respect to the non-relativistic case. This indicates that in a medium with a stiffer EOS the energy propagation through solitary waves is more efficient.

It is interesting to observe the supersonic nature of the solutions (45) and (47), which is manifest in the arguments of the square roots. As a final remark about (43) we notice that, for

$$\frac{\Phi \rho_0^2}{2\mu_B c_s^2} < c_s^2 - \frac{3}{2} \quad (48)$$

the solution (43) becomes negative and, in view of (36), can be interpreted as a rarefaction wave. A solution of this type was found in [9] where nuclear matter was described by an EOS based on the Skyrme force.

V. CONCLUSIONS

The existence of KdV solitons in nuclear matter has potential applications in nuclear physics at intermediate energies [1] and also possibly at high energies. The experimental

measurements of jet quenching and related phenomena performed at RHIC [12] offer an unique opportunity of studying supersonic motion in hot and dense hadronic matter. With this scenario in mind we took the first steps in the adaptation of the KdV soliton formalism to the new environment. We have extended the results of our previous work [2], showing that it is possible to obtain the KdV solitons in relativistic hydrodynamics. Moreover we have explored other equations of state. Taking the non-relativistic limit ($c_s^2 \rightarrow 0$) we were able to recover the previous results.

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